Estimating the rate of successful Diophantine approximations

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Abstract

Diophantine approximation studies how well real numbers can be approximated by rational numbers. Previous theorems, such as Khintchine's theorem and the Duffin-Schaeffer conjecture, has implied that the inequalities $|x - \frac{p}{q}| < \frac{\varepsilon}{q^2}$, with $\frac{p}{q}$ being Farey fractions, can accept infinitely many solutions for almost every x within the interval $[0,1]$. Our project studies a pertinent and interesting extension of this statement.

We intended to find the number of solutions of such inequalities with the denominator $q < Q$ for a fixed Q. Our conjecture is that under the limit when Q goes to infinity, the ratio between the number of solutions and the summation of the lengths of the intervals $[\frac{p}{q} - \frac{\varepsilon}{q^2}, \frac{p}{q} + \frac{\varepsilon}{q^2}]$ should converge to 1. Though we haven't proven it yet, we've managed to find an upper bound and a lower bound for this ratio using the method of continued fractions when $\varepsilon < \frac{1}{2}$. The fact that these bounds are so close to 1 further supported the validity of our conjecture. Using a probabilistic argument, we also proved that the limit is indeed 1 when the fraction $\frac{p}{q}$ only takes prime denominators.

Introduction

Not all real numbers can be expressed using fractions. For any irrational number α, we can only approximate it by a certain fraction $\frac{p}{q}$ with an inaccuracy $|\alpha - \frac{p}{q}|$. Thus, how well we are able to approximate α using various $\frac{p}{q}$'s becomes an area of interest for the mathematicians. Particularly, they are interested in the inequalities $|\alpha - \frac{p}{q}| < f(q)$ for various functions $f\colon \mathbb{N} \to \mathbb{R}$.

One of the most famous conjectures in the field, called the Duffin-Schaeffer conjecture, has just been proven recently and stated $|\alpha - \frac{p}{q}| < \frac{\psi(q)}{q}$ is satisfied infinitely many times for $\gcd(p,q) = 1$ and almost every $\alpha \in [0,1]$ when $\sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} = \infty$, where $\varphi(q)$ is the Euler's totient function counting the number of integers $p \leqslant q$ such that $\gcd(p,q) = 1$. This theorem implied that $|\alpha - \frac{p}{q}| < \frac{\varepsilon}{q^2}$ accepts an infinite number of solutions when $\gcd(p,q) = 1$ for almost every α.

As an extension to this claim, we want to find out how many times this inequality is satisfied when $q < Q$ by understanding the asymptotic behavior when $Q \to \infty$. One immediate guess might be that for almost every $\alpha \in [0,1]$,

$$\lim_{Q\to\infty} \frac{\sum_{q\leqslant Q, \gcd(p,q)=1} \chi_{B(\frac{p}{q},\frac{\varepsilon}{q^2})}(\alpha)}{\sum_{q=1}^{Q} \frac{2\varepsilon\varphi(q)}{q^2}} = 1, \tag{1}$$

which hopes for some "independence behavior" (analogy from probability) that causes the summation of the characteristic functions χ applied on α tend to the summation of their expected values.

Methods

Continued Fractions

A classical method used in treating Diophantine approximation problems is the continued fractions. Continued fraction, as the name suggests, is an expansion of any positive real number into the following form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}; \qquad \alpha = [a_0; a_1, a_2 \cdots], \tag{2}$$

where each a_n takes only positive integer values. If α is irrational, this expansion will continue infinitely, but we can always approximate it only using a finite expansion of continued fractions. The fraction

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots, a_n] \tag{3}$$

is called the n^{th} convergent of α.

Such expansion turns out to have a close relationship with the *best approximations of the second kind*. A fraction $\frac{a}{b}$ is called a best approximation of the second kind of α if whenever $\frac{c}{d} \neq \frac{a}{b}$ and $0 < d \leqslant b$, we have $|c\alpha - d| > |b\alpha - a|$. In fact, there exists an inconspicuous yet amazing bijection between the best approximations and the convergents, which makes the continued fractions exceptionally useful in such approximations.

An Analogy of the Law of Large Numbers

Another proposition turned out to be important to our research:

Let $H_i\colon [0,1] \to \mathbb{R}_{>0}$. Suppose there exists C_1 and C_2 such that for all i,
(H1) $\|H_i\|_\infty < C_1$,
(H2) $\sum_{i=1}^{\infty} \|H_i\|_1 = \infty$,
(H3) $\sum_{j=i+1}^{\infty} |\int H_i H_j - \int H_i \int H_j| < C_2 \|H_{i-1}\|_1$,
then we have for almost all $x \in [0,1]$,

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} H_i(x)}{\sum_{i=1}^{N} \int H_i} = 1. \tag{4}$$

This statement is analogous to the Law of Large Numbers in that (H1) and (H2) modeled the "identically distributed" condition and (H3) mimicked the "independently distributed" condition.

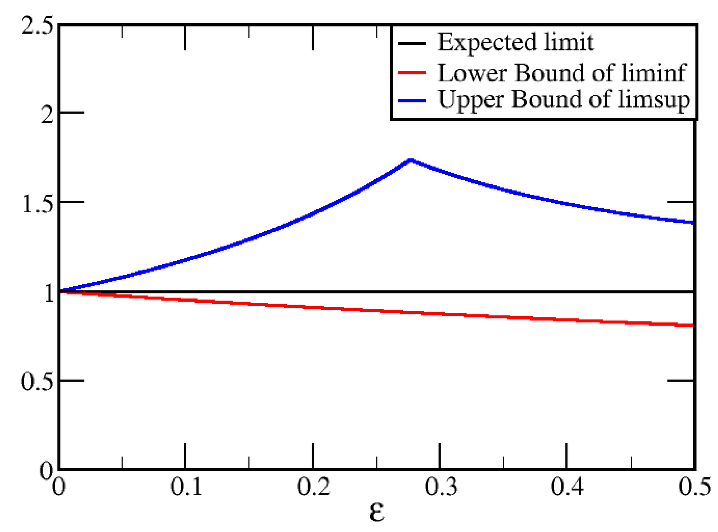
Results

Every irreducible rational fraction $\frac{p}{q}$ that satisfies $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$ is a convergent of α. This fact indicated that it will be very beneficial to count the number of convergents with denominators $q_n < Q$ in the case when $\varepsilon < \frac{1}{2}$. With this insight, we obtained upper bounds and a lower bound for the left hand side limit of *eq.*(1):

$$\liminf_{Q\to\infty} \frac{\sum_{q\leqslant Q, \gcd(p,q)=1} \chi_{B(\frac{p}{q},\frac{\varepsilon}{q^2})}(\alpha)}{\sum_{q=1}^{Q} \frac{2\varepsilon\varphi(q)}{q^2}} > \frac{\ln(1+\varepsilon)}{\varepsilon} \text{ for } \varepsilon < \frac{1}{2}; \tag{5}$$

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q, \gcd(p,q)=1} \chi_{B(\frac{p}{q},\frac{\varepsilon}{q^2})}(\alpha)}{\sum_{q=1}^{Q} \frac{2\varepsilon\varphi(q)}{q^2}} < \frac{\ln(\frac{1-\varepsilon}{1-2\varepsilon})}{\varepsilon} \text{ for } \varepsilon < \frac{1}{3}; \tag{6}$$

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q, \gcd(p,q)=1} \chi_{B(\frac{p}{q},\frac{\varepsilon}{q^2})}(\alpha)}{\sum_{q=1}^{Q} \frac{2\varepsilon\varphi(q)}{q^2}} < \frac{\ln(1+\sqrt{\frac{\varepsilon}{1-\varepsilon}})}{\varepsilon} \text{ for } \varepsilon < \frac{1}{2}. \tag{7}$$



Using the probabilistic argument above, we also managed to prove that when the denominators q takes only prime numbers, for any value of ε,

$$\lim_{Q\to\infty} \frac{\sum_{q\leqslant Q, q \text{ prime}} \chi_{B(\frac{p}{q},\frac{\varepsilon}{q^2})}(\alpha)}{\sum_{q\leqslant Q, q \text{ prime}} \frac{2\varepsilon\varphi(q)}{q^2}} = 1. \tag{8}$$

Future Works

Currently, we are in search for a dense subset of $[0,1]$ where *eq.*(1) is satisfied. We have created a potential recipe for constructing such numbers, and we might be able to measure the Hausdorff dimension of this subset. We are aware that some other statements in continued fractions can lead to *eq.*(1), and we will also attempt to prove that *eq.*(1) is indeed true for almost every number in $[0,1]$.

References

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[2] A. Ya. Khinchin. *Continued Fractions*. Dover Publications, 3rd edition, 1997.