Differential Geometry: Riemannian, Contact, Symplectic

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Abstract:

Our objectives were:

(1) to develop a deeper understanding of the three branches of differential geometry, which are Riemannian, Contact, and Symplectic;

(2) to study the applications of the theoretical concepts to Physics and other mathematical areas.

No original results will be presented. We summarize the material we studied.

Introduction:

Geometry is a set with a structure tensor defined on it. The structure tensor is a function, which is defined differently in disparate branches. In particular:

Riemannian geometry: a metric tensor (associates a real number to a pair of tangent vectors from a tangent space at each point) which is symmetric and positive-definite; used to define such concepts as *length of a curve*, *angle*, *distance*, *volume*, etc.

Contact geometry: a contact form which is a differential 1-form with stipulation that it is non-degenerate; the *contact distribution* (kernel of the contact form) is the central object of study.

Symplectic geometry: a symplectic form is a differential 2-form satisfying the properties of being non-degenerate and closed; the central object of study is the *symplectic form* itself.

Riemannian Geometry:

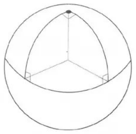
Riemannian geometry generalizes the Euclidean geometry we are familiar with to other spaces, such as the surface of a sphere. The Riemannian metric gives us a notion of distance between vectors. This allows us to define geometric properties we are already familiar with, such as length and angle on more exotic spaces.

Geodesics:

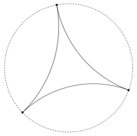
In Euclidean space, the shortest path between any two points is always the straight line between them. To find shortest paths, or *geodesics*, in a general Riemannian space, we find curves in the space whose tangent vectors don't change with respect to the curve itself: ▽_{c'} c' = 0 .

Curvature:

Applying geometry to more general spaces also gives rise to a notion of *curvature*. One variant, called *sectional curvature*, describes curvature per unit area with respect to the metric. Euclidean space has 0 curvature. Curvature has geometric consequences on spaces, including on the shape of triangles. The sphere and hyperbolic plane have constant sectional curvatures of 1 and -1 respectively.



← The sphere has positive curvature, so triangles can be formed with all right angles.



← The hyperbolic plane is negatively curved, so triangles can be formed with much smaller angles.

Summary of the two applications of Contact Geometry:

(1) To Physics: Huygen's Principle describes the evolution of the wavefront, which in mathematical language is expressed as a diffeomorphism (a map that "preserves structure"). The evolution of the wavefront has the property that a point ***q*** on the wavefront ***i*** gets mapped to a point ***q'*** on the wavefront ***i+1*** such that the tangent line to the wavefront at ***q*** coincides with the tangent line at ***q'***. To record the information about the tangent line, the space of contact elements is introduced. It consists of points **(x,y,z)** in **R^3** with the meaning of the third coordinate changed, - it is the slope of a tangent line to the wavefront at a point **(x,y)**. A diffeomorphism is then defined on the contact space in which a wavefront is modelled.

(2) To Math: An ordinary differential equation (ODE) consists of an unknown function, its derivative, and an independent variable. The solution to ODE is a function. We represent ODE as a function **F** in the contact space. The solution to it is identified with the zero level set of **F**. Although this does not give an "easier" way to solve an ODE, the method is very powerful when applied to solving partial differential equations in higher dimensions.

Application of Symplectic Geometry to Hamiltonian Mechanics:

Newtonian mechanics have the issue of second order systems of differential equations becoming very difficult to solve as the systems become more complex. Hamiltonian mechanics seeks to amend this by recasting the problem in terms of momentum instead of in terms of force. Given a hamiltonian function H, there is a unique vector field X_H defined i(X_H) w_0 = H, where w_0 is the standard symplectic form on R^2. The integral curves of the vector field X_H give the solution to the system defined by the hamiltonian function H. The solution to H then defines the motion of the system.

Reference: *McInerney, A. (2013). First Steps in Differential Geometry: Riemannian, Contact, Symplectic (Undergraduate Texts in Mathematics). Springer.*